

Contact 3-manifolds and Ricci solitons

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Abstract

A contact 3-manifold M admitting a transversal Ricci soliton (g, v, λ) is either Sasakian or locally isometric to one of the Lie groups $SU(2)$, $SL(2, R)$, $E(2)$, $E(1, 1)$ with a left invariant metric.

1 Introduction

Contact geometry is motivated by classical mechanics. Where a symplectic space is considered the even-dimensional phase space of a mechanical system, a contact space corresponds to the odd-dimensional extended phase space that includes the time variable. A *contact manifold* (M, η) is a smooth manifold M^{2n+1} together with a global one-form η such that $d\eta$ has maximal rank $2n$ on the contact distribution $D = \ker \eta$. The duality of η defines a unique vector field ξ , the *Reeb vector field*. The *Reeb flow* is a one parameter group of diffeomorphisms $\{\phi_t\}$ generated by the Reeb vector field ξ . Martinet [11] proved that every closed, orientable 3-manifold admits a contact structure.

A *Ricci soliton* is defined on a Riemannian manifold (M, g) by

$$(1) \quad \frac{1}{2} \mathcal{L}_W g + \text{Ric} - \lambda g = 0$$

where W is a vector field (the potential vector field), λ a constant on M . A Ricci soliton with W zero is reduced to Einstein equation. Not only for studying the topology of manifolds, but in the study of string theory, theoretical physicists also have been looking into the equation of Ricci solitons [8]. Compact Ricci solitons are the fixed points of the *Ricci flow*:

$$(2) \quad \frac{\partial}{\partial t} g = -2 \text{Ric}$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds.

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Indeed, the Ricci flow $\{g_t\}$ is equivalent to the initial metric $g = g_0$ satisfying the Ricci soliton equation (1) for some vector field W and some constant λ . The Ricci soliton is said to be shrinking, steady, and expanding according as $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$ respectively. For details we refer to [4] or [6] about the Ricci flows and their solitons. Hamilton [10] initiated the Ricci flow theory, namely, he proved that on a compact manifold M , the Ricci flow equation with any prescribed initial metric g_0 has a unique solution on some maximal time interval $[0, T)$, where $0 < T \leq \infty$. Developing it, Hamilton proved that a compact 3-manifold admitting a metric of strictly positive Ricci curvature admits in fact a metric of constant positive sectional curvature. In particular, if such a manifold is simply connected, the manifold is isometric to the sphere, and then is diffeomorphic to the S^3 . Then the Ricci flow has been a very attractive approach to a possible positive answer to the Poincaré Conjecture [13], more generally to the Thurston's Geometrization Conjecture [14], which describes all 3-dimensional manifolds in terms of the Eight Geometries (cf. Remark 2).

Now, we consider the Ricci flow in contact geometry [5]. Then, we keeping Martinet's result in mind, it is very natural and interesting to study the Ricci flow in contact 3-manifolds. In this context, we first establish a very special Ricci flow which evolves by the Reeb flow and a (time dependent) evolving factor at the same time. Then we have the corresponding Ricci soliton equation, which we call a *contact Ricci soliton*:

$$(3) \quad \frac{1}{2} \mathcal{L}_\xi g + \text{Ric} - \lambda g = 0.$$

In [5], we proved

Theorem 1. *A 3-dimensional contact Ricci soliton (g, ξ, λ) is of constant curvature +1.*

Next, as a complementary partner of the contact Ricci soliton, we consider the so-called *transversal Ricci soliton*:

$$(4) \quad \frac{1}{2} \mathcal{L}_v g + \text{Ric} - \lambda g = 0,$$

where v is a complete vector field orthogonal to ξ . Then, we prove in Section 3 the following theorem.

Theorem 2. *A contact 3-manifold M admitting a transversal Ricci soliton (g, v, λ) is either Sasakian or locally isometric to one of the following Lie groups with a left invariant metric: $SU(2)$, $SL(2, R)$, $E(2)$ (the group of rigid motions of the Euclidean 2-space), $E(1, 1)$ (the group of rigid motions of the Minkowski 2-space).*

2 Contact geometry and the known results

We start by reviewing briefly the fundamental materials about contact geometry. All manifolds in the present paper are assumed to be connected, oriented and smooth.

A $(2n + 1)$ -dimensional manifold M is a *contact manifold* if it is equipped with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η , there exists a unique vector field ξ , called the *Reeb vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X . It is well-known that there also exists a Riemannian metric g and a $(1, 1)$ -tensor field φ such that

$$(5) \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi,$$

where X and Y are vector fields on M . From (5), it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$. A Riemannian manifold M equipped with structure tensors (η, g) satisfying (5) is said to be a *contact Riemannian manifold* or a *contact metric manifold* and it is denoted by $M = (M, \eta, g)$. Given a contact Riemannian manifold M , we define a $(1, 1)$ -tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi\varphi$. Then h is self-adjoint and satisfies

$$(6) \quad h\xi = 0, \quad h\varphi = -\varphi h,$$

$$(7) \quad \nabla_X \xi = -\varphi X - AX,$$

where ∇ is Levi-Civita connection and $A = \varphi h$. From (6) and (7), we see that each trajectory of ξ is a geodesic flow. Moreover, we also have

$$(8) \quad \text{Ric}(\xi, \xi) = 2n - \text{trace } h^2$$

(cf. Corollary 7.1 in [1]). A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold*. It is easy to see that a contact Riemannian manifold is *K-contact* if and only if $h = 0$. For a contact manifold M , the tangent space $T_p M$ of M at each point $p \in M$ is decomposed as $T_p M = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_p M | \eta(v) = 0\}$. Then the $2n$ -dimensional distribution (or subbundle) $D : p \rightarrow D_p$ is called the *contact distribution (or contact subbundle)*. For a contact manifold M , the associated almost CR structure is given by the holomorphic subbundle $\mathcal{H} = \{X - iJX : X \in D\}$ of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM , where $J = \varphi|_D$, the restriction of φ to D . We say that *the almost CR structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. A *Sasakian manifold* is a *K-contact manifold* whose associated almost CR structure is integrable. Then we observe that a 3-dimensional *K-contact manifold* is already Sasakian.

Recall another class of contact metric manifolds, the so-called *contact (α, β) -manifolds* (introduced by Blair, Koufogiorgos and Papantoniou [3]) are defined by the curvature condition

$$R(X, Y)\xi = \alpha(\eta(Y)X - \eta(X)Y) + \beta(\eta(Y)hX - \eta(X)hY)$$

for arbitrary vector fields X, Y and for some real numbers α and β . The class of contact (α, β) -manifolds is developed from a contact metric manifold with $R(X, Y)\xi = 0$. Indeed, by a *D-homothetic deformation* of a contact metric manifold with $R(X, Y)\xi =$

0, we obtain a contact (α, β) -manifold. Here, for a positive constant ϵ , D_ϵ -homothetic deformation means a change of structure tensors by

$$\bar{\eta} = \epsilon\eta, \quad \bar{\xi} = \frac{1}{\epsilon}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = \epsilon g + \epsilon(\epsilon - 1)\eta \otimes \eta.$$

This class includes Sasakian manifolds (for $\alpha = 1$ and $h = 0$) and the trivial sphere bundle $E^{n+1} \times S^n$ (for $\alpha = \beta = 0$). Characteristic examples of non-Sasakian (α, β) -contact spaces are the tangent sphere bundles of Riemannian manifolds of constant curvature $\neq 1$. For the three dimensions, such spaces are classified in [3].

Theorem 3. *A 3-dimensional contact (α, β) -space is either Sasakian or locally isometric to one of the Lie groups $SU(2)$, $SL(2, R)$, $E(2)$, $E(1, 1)$ with a left invariant metric.*

Boeckx and the present author [2] proved the following theorem, which has a crucial role in proving our main Theorem 2.

Theorem 4. *A 3-dimensional contact manifold with η -parallel h , which means $g((\nabla_x h)y, z) = 0$ for any vector fields x, y, z orthogonal to ξ , is a contact (α, β) -space.*

We refer to [1] for the above formulas, results and the further details on contact Riemannian geometry.

3 Proof of Theorem 2

We consider on M the maximal open subset \mathcal{U}_1 on which $h \neq 0$ and the maximal open subset \mathcal{U}_2 on which h is identically zero. Then $\mathcal{U}_1 \cup \mathcal{U}_2$ is open dense in M . Suppose that M is non-Sasakian. Then \mathcal{U}_1 is non-empty and there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ such that $he_1 = \mu e_1$, $he_2 = -\mu e_2$, where μ is a smooth function. First of all, we prepare the following lemma.

Lemma 5. (cf. [7]) *Let M be a 3-dimensional contact Riemannian manifold. Then with respect to \mathcal{E} , the Levi-Civita connection ∇ is given by*

$$\begin{aligned} \nabla_{e_1} e_1 &= be_2, & \nabla_{e_1} e_2 &= -be_1 + (1 + \mu)\xi, & \nabla_{e_1} \xi &= -(1 + \mu)e_2, \\ \nabla_{e_2} e_1 &= -ce_2 + (\mu - 1)e_3, & \nabla_{e_2} e_2 &= ce_1, & \nabla_{e_2} \xi &= (1 - \mu)e_1, \\ \nabla_\xi e_1 &= ae_2, & \nabla_\xi e_2 &= -ae_1, & \nabla_\xi \xi &= 0, \end{aligned}$$

where a, b, c are smooth functions. The Ricci operator S is given by

$$\begin{aligned} Se_1 &= \text{Ric}(e_1, e_1)e_1 + \xi(\mu)e_2 + (2b\mu - e_2(\mu))\xi, \\ Se_2 &= \xi(\mu)e_1 + \text{Ric}(e_2, e_2)e_2 + (2c\mu - e_1(\mu))\xi, \\ S\xi &= (2b\mu - e_2(\mu))e_1 + (2c\mu - e_1(\mu))e_2 + 2(1 - \mu^2)\xi. \end{aligned}$$

For \mathcal{E} , it is known that $e_3 = \xi$ is defined globally on M . Lifting to the universal covering space \widetilde{M}^3 if necessary we have global orthonormal frame field, which also denoted by e_1, e_2 and e_3 . Now, we assume that M^3 admits a transversal Ricci soliton (g, v, λ) with $v = f_1 e_1 + f_2 e_2$, where f_1 and f_2 are smooth functions. Then the Ricci soliton equation (4) is written by

$$(9) \quad \frac{1}{2} \left(g(\nabla_X v, Y) + g(\nabla_Y v, X) \right) + \text{Ric}(X, Y) - \lambda g(X, Y) = 0.$$

We compute

$$\nabla_X v = (X f_1) e_1 + f_1 \nabla_X e_1 + (X f_2) e_2 + f_2 \nabla_X e_2.$$

Putting $X = Y = \xi$ in (9), then by using (8) we get $\lambda = 2 - 2\mu^2$, from which we find that μ is constant. Put $X = Y = e_1$ to get

$$(10) \quad e_1(f_1) - b f_2 + \text{Ric}(e_1, e_1) = \lambda.$$

Putting $X = Y = e_2$, then we obtain

$$(11) \quad e_2(f_2) - c f_1 + \text{Ric}(e_2, e_2) = \lambda.$$

If we put $X = \xi$ and $Y = e_1$ in (9), then using the formula in Lemma 5 we get

$$(12) \quad \frac{1}{2} \left(\xi(f_1) - a f_2 + (1 + \mu) f_2 \right) + \text{Ric}(\xi, e_1) = 0.$$

Put $X = \xi$ and $Y = e_2$ in (9) to get

$$(13) \quad \frac{1}{2} \left(a f_1 + \xi(f_2) + (\mu - 1) f_1 \right) + \text{Ric}(\xi, e_2) = 0.$$

Since $\text{Ric}(e_1, e_2) = 0$, if we put $X = e_1$ and $Y = e_2$ in (9), then we obtain

$$(14) \quad b f_1 + e_1(f_2) + e_2(f_1) + c f_2 = 0.$$

We suppose that f_1 and f_2 are pointwise linearly independent functions. Give an additional assumption:

$$(15) \quad e_1(f_2) + e_2(f_1) = 0.$$

Then, from (14) we have $b = c = 0$. From the computation $(\nabla_{e_i} h) e_j = \nabla_{e_i} (h e_j) - h \nabla_{e_i} e_j$ for $i, j = 1, 2$, we can easily obtain $g((\nabla_{e_i} h) e_j, e_j) = 0$, $g((\nabla_{e_1} h) e_1, e_2) = 2\mu g(\nabla_{e_1} e_1, e_2) = 0$ and $g((\nabla_{e_2} h) e_1, e_2) = 2\mu g(\nabla_{e_2} e_1, e_2) = 0$, where we have used μ is constant and $b = c = 0$. Namely, we have

Lemma 6. *h is η -parallel, i.e. $g((\nabla_x h) y, z) = 0$ for any vector fields x, y, z orthogonal to ξ .*

Hence, from Theorem 4 we have M is a contact (α, β) -space. Then, due to Theorem 3 we have M is locally isometric to one of the following Lie groups with a left invariant metric: $SU(2)$, $SL(2, R)$, $E(2)$, $E(1, 1)$.

Then, from now we find their associated potential vector fields v explicitly. For the non-Sasakian (α, β) -manifold, we already know $a = -\beta/2$ (constant) (Lemma 4.1 in [3]) and the Ricci operator S is given by

$$(16) \quad S = -\beta I + \beta h + (2\alpha + \beta)\eta \otimes \xi,$$

where I denotes the identity transformation (Remark 3.2 in [3]). Thus, from (10), (11), (12), (13), (15), and (16) we have

$$(17) \quad \begin{cases} e_1(f_2) + e_2(f_1) = 0, \\ e_1(f_1) - \delta_1 = 0, \\ e_2(f_2) - \delta_2 = 0, \\ \xi(f_1) + \delta_3 f_2 = 0, \\ \xi(f_2) + \delta_4 f_1 = 0, \end{cases}$$

where $\delta_1 = \beta - \beta\mu + 2 - 2\mu^2$, $\delta_2 = \beta + \beta\mu + 2 - 2\mu^2$, $\delta_3 = \beta/2 + 1 + \mu$, and $\delta_4 = -\beta/2 + \mu - 1$.

Moreover, we also observe that e_1, e_2, ξ are geodesic vector fields. We adapt a normal coordinate system (u^1, u^2, t) at $p \in M$, i.e. $\exp_p(\sum_{i=1}^2 u^i(q)e_i + t(q)\xi) = q$. From the last two equations of (17), we establish 2nd order differential equations of constant coefficients:

$$(18) \quad \frac{\partial^2 f_i}{\partial t^2} - \delta f_i = 0$$

at p , for $i = 1, 2$, where we have put $\delta = \delta_3\delta_4$. Then we obtain their suitable solutions for $\delta > 0$, $\delta < 0$, $\delta = 0$, respectively.

(I) For $\delta > 0$, we first get a general solution form of (18):

$$\begin{cases} f_1(u^1, u^2, t) = A_1(u^1, u^2) \exp(\sqrt{\delta}t) + B_1(u^1, u^2) \exp(-\sqrt{\delta}t), \\ f_2(u^1, u^2, t) = A_2(u^1, u^2) \exp(\sqrt{\delta}t) + B_2(u^1, u^2) \exp(-\sqrt{\delta}t), \end{cases}$$

where $A_i(u^1, u^2), B_i(u^1, u^2)$, $i = 1, 2$, are smooth functions for u^1, u^2 . Further, from the first three equations of (17) we may take

$$(19) \quad \begin{cases} A_1(u^1, u^2) = (\delta_1/2)u^1 + (\delta_1/2)u^2 + \tilde{C}_1, \\ B_1(u^1, u^2) = (\delta_1/2)u^1 + (\delta_1/2)u^2 + \tilde{D}_1, \\ A_2(u^1, u^2) = (-\delta_1/2)u^1 + (\delta_2/2)u^2 + \tilde{C}_2, \\ B_2(u^1, u^2) = (-\delta_1/2)u^1 + (\delta_2/2)u^2 + \tilde{D}_2. \end{cases}$$

Finally, reflecting the last two equations of (17) again, we obtain

$$(20) \quad \begin{cases} \tilde{C}_1 = -\sqrt{\delta_3}C, \tilde{D}_1 = \sqrt{\delta_3}D, \tilde{C}_2 = \sqrt{\delta_4}C, \tilde{D}_2 = \sqrt{\delta_4}D, \text{ for } \delta_3 > 0, \delta_4 > 0, \\ \tilde{C}_1 = \sqrt{|\delta_3|}C, \tilde{D}_1 = -\sqrt{|\delta_3|}D, \tilde{C}_2 = \sqrt{|\delta_4|}C, \tilde{D}_2 = \sqrt{|\delta_4|}D, \text{ for } \delta_3 < 0, \delta_4 < 0. \end{cases}$$

where C, D are arbitrary constants. In similar ways, we have

(II) For $\delta < 0$;

$$\begin{cases} f_1(u^1, u^2, t) = A_1(u^1, u^2) \cos(\sqrt{-\delta}t) + B_1(u^1, u^2) \sin(\sqrt{-\delta}t), \\ f_2(u^1, u^2, t) = A_2(u^1, u^2) \cos(\sqrt{-\delta}t) + B_2(u^1, u^2) \sin(\sqrt{-\delta}t). \end{cases}$$

Here,

$$(21) \quad \begin{cases} A_1(u^1, u^2) = \delta_1 u^1 + \delta_1 u^2 + \tilde{C}_1, \\ A_2(u^1, u^2) = -\delta_1 u^1 + \delta_2 u^2 + \tilde{C}_2, \\ B_1(u^1, u^2) = \delta_1 u^1 + \delta_1 u^2 + \tilde{D}_1, \\ B_2(u^1, u^2) = -\delta_1 u^1 + \delta_2 u^2 + \tilde{D}_2, \end{cases}$$

and

$$(22) \quad \begin{cases} \tilde{C}_1 = -\sqrt{\delta_3}C, \tilde{D}_1 = \sqrt{\delta_3}D, \tilde{C}_2 = -\sqrt{|\delta_4|}D, \tilde{D}_2 = -\sqrt{|\delta_4|}C, \text{ for } \delta_3 > 0, \delta_4 < 0, \\ \tilde{C}_1 = \sqrt{|\delta_3|}C, \tilde{D}_1 = \sqrt{|\delta_3|}D, \tilde{C}_2 = \sqrt{\delta_4}D, \tilde{D}_2 = -\sqrt{\delta_4}C, \text{ for } \delta_3 < 0, \delta_4 > 0, \end{cases}$$

where C, D are arbitrary constants.

(III) For $\delta = 0$;

$$\begin{cases} f_1(u^1, u^2, t) = A_1(u^1, u^2) + B_1(u^1, u^2)t, \\ f_2(u^1, u^2, t) = A_2(u^1, u^2) + B_2(u^1, u^2)t, \end{cases}$$

where

$$(23) \quad \begin{cases} A_1(u^1, u^2) = \delta_1 u^1 + C, \\ A_2(u^1, u^2) = \delta_2 u^2 + D, \\ B_1(u^1, u^2) = -\delta_3 D, \\ B_2(u^1, u^2) = -\delta_4 C, \end{cases}$$

and C, D are arbitrary constants. (For a locally flat manifold ($\mu = 1, \beta = 0$), we find $f_1 = C - 2Dt$ and $f_2 = D$.)

Then, from the relations (19) and (20) for $\delta > 0$ ((21) and (22) for $\delta < 0$, (23) for $\delta = 0$, respectively) we can actually show that $bf_1 + cf_2 = 0$ implies $b = c = 0$ at p . For example, for the case $\delta_3 > 0, \delta_4 > 0$ we compute $bf_1 + cf_2 = 0$ at p : $-b(p)\sqrt{\delta_3}(C - D) + c(p)\sqrt{\delta_4}(C + D) = 0$ for arbitrary constants C and D . Then, it implies at once $b = c = 0$ since $\delta (= \delta_3\delta_4) \neq 0$. After all, $v = f_1e_1 + f_2e_2$ are the desired potential vector fields. This completes the proof of Theorem 2.

We describe the Ricci solitons for the special cases $\beta = 0$ by the table below.

The special cases ($\beta = 0$).

	$0 < \mu < 1$	$\mu > 1$	$\mu = 1$
δ	negative	positive	zero
Potential vector field type	II	I	III
Ricci soliton	shrinking	expanding	steady
Lie group	SU(2)	SL(2, R)	E(2)

4 The Sasakian case

In the remainder of this paper, we treat a Ricci soliton for Sasakian 3-manifolds. Geiges [9] proved the following result.

Theorem 7. *A closed 3-manifold admits a Sasakian structure if and only if it is diffeomorphic to a quotient of one of the spaces S^3 , $\widehat{SL(2, R)}$ or Nil by a discrete group of fixed point free isometries.*

Let M be a 3-dimensional unimodular Lie group with a left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$. Then, according to a result of Perrone [12] M admits its compatible left invariant Sasakian structure if and only if there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra \mathfrak{m} such that

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_1 e_2.$$

The Reeb vector field ξ is obtained by left translation of e_3 . The contact distribution D is spanned by e_1 and e_2 . It includes $SU(2)$, $SL(2, R)$, Nil (Heisenberg group), for c_1 positive, negative, zero, respectively.

By the Koszul formula, one can calculate the Levi-Civita connection ∇ in terms of the basis $\{e_1, e_2, e_3 = \xi\}$ as follows:

$$(24) \quad \begin{aligned} \nabla_{e_1} e_1 &= 0, & \nabla_{e_1} e_2 &= e_3, & \nabla_{e_1} e_3 &= -e_2, \\ \nabla_{e_2} e_1 &= -e_3, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_3 &= e_1, \\ \nabla_{e_3} e_1 &= (c_1 - 1)e_2, & \nabla_{e_3} e_2 &= -(c_1 - 1)e_1, & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

From the above data, we are already aware that e_1, e_2, e_3 are geodesic vector fields. Moreover, we find that the basis $\{e_1, e_2, e_3 = \xi\}$ diagonalizes the Ricci operator, and we have

$$(25) \quad \text{Ric}(e_1, e_1) = \text{Ric}(e_2, e_2) = 2c_1 - 2, \quad \text{Ric}(\xi, \xi) = 2.$$

Namely, the Ricci operator S is represented by $S = (2c_1 - 2)I + (4 - 2c_1)\eta \otimes \xi$, which says that M is η -Einstein. We suppose that M admits a transversal Ricci soliton (g, v, λ) with $v = f_1 e_1 + f_2 e_2$, where f_1 and f_2 are smooth functions. Then we have

$$(26) \quad \frac{1}{2} \left(g(\nabla_X v, Y) + g(\nabla_Y v, X) \right) + \text{Ric}(X, Y) - \lambda g(X, Y) = 0.$$

By similar computations in section 3, using (24), (25) and (26), we finally obtain $\lambda = 2$ (a shrinking Ricci soliton) and

$$(27) \quad \begin{cases} e_1(f_2) + e_2(f_1) = 0, \\ e_1(f_1) = e_2(f_2) = 4 - 2c_1, \\ \xi(f_1) - (c_1 - 2)f_2 = 0, \\ \xi(f_2) + (c_1 - 2)f_1 = 0. \end{cases}$$

From the last two equations of (27) we have

$$\frac{\partial^2 f_i}{\partial t^2} + (c_1 - 2)^2 f_i = 0$$

for $i = 1, 2$. Then, adapting a normal coordinate system (u^1, u^2, t) associated with $\{e_1, e_2, \xi\}$ and using similar arguments in section 3, we have a special solution for $c_1 \neq 2$:

$$\begin{cases} f_1(u^1, u^2, t) = A_1(u^1, u^2) \cos(|c_1 - 2|t) + B_1(u^1, u^2) \sin(|c_1 - 2|t), \\ f_2(u^1, u^2, t) = A_2(u^1, u^2) \cos(|c_1 - 2|t) + B_2(u^1, u^2) \sin(|c_1 - 2|t). \end{cases}$$

Here,

$$\begin{cases} A_1(u^1, u^2) = (4 - 2c_1)(u^1 + u^2) - D, \\ A_2(u^1, u^2) = (4 - 2c_1)(-u^1 + u^2) + C, \\ B_1(u^1, u^2) = (4 - 2c_1)(u^1 + u^2) + C, \\ B_2(u^1, u^2) = (4 - 2c_1)(-u^1 + u^2) + D, \end{cases}$$

for $c_1 > 2$, or

$$\begin{cases} A_1(u^1, u^2) = (4 - 2c_1)(u^1 + u^2) - D, \\ A_2(u^1, u^2) = (4 - 2c_1)(-u^1 + u^2) - C, \\ B_1(u^1, u^2) = (4 - 2c_1)(u^1 + u^2) + C, \\ B_2(u^1, u^2) = (4 - 2c_1)(-u^1 + u^2) - D, \end{cases}$$

for $c_1 < 2$, where C, D are arbitrary constants. For the case $c_1 = 2$ (M the unit sphere), we get a special $v = Ce_1 + De_2$, where C, D are arbitrary constants.

For the Heisenberg group, we write down the Ricci soliton (g, v, λ) in more detail.

Example (Heisenberg group). For the Heisenberg group

$$\mathbb{H} = \left\{ \left(\begin{pmatrix} 1 & u^2 & t \\ 0 & 1 & u^1 \\ 0 & 0 & 1 \end{pmatrix} \middle| u^1, u^2, t \in R \right\},$$

the contact form is $\eta = 1/2(dt - u^2 du^1)$ and the Reeb vector field is $\xi = 2\frac{\partial}{\partial t}$. The Riemannian metric $ds^2 = \eta \otimes \eta + 1/4((du^1)^2 + (du^2)^2)$ is a left invariant metric and it is a Sasakian metric associated with η (cf. Example 4.5.1 in [1]). For the Heisenberg group \mathbb{H} ($c_1 = 0$), we get a potential vector field $v = f_1 e_1 + f_2 e_2$, where

$$\begin{cases} f_1(u^1, u^2, t) = (4(u^1 + u^2) - D) \cos(2t) + (4(u^1 + u^2) + C) \sin(2t), \\ f_2(u^1, u^2, t) = (4(-u^1 + u^2) - C) \cos(2t) + (4(-u^1 + u^2) - D) \sin(2t), \end{cases}$$

for arbitrary constants C, D .

We close this paper by mentioning the Eight Geometries.

Remark (Thurston geometry). The eight model spaces of Thurston geometry are 3-dimensional space forms S^3, R^3, H^3 ; product spaces $S^2 \times R$ and $H^2 \times R$;

Nil and $\widetilde{SL(2, R)}$; Sol. (Class $E(1.1)$ corresponds to Sol and $E(2)$ belongs to the flat geometry R^3 .)

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